Truly-mixed finite-elements for the analysis of viscoelastic devices

Paolo Calvi\textsuperscript{1}, Marco Pingaro\textsuperscript{2}, Paolo Venini\textsuperscript{2}
\textsuperscript{1}\textit{Department of Civil Engineering, University of Toronto, Canada}
E-mail: paolo.calvi@utoronto.ca
\textsuperscript{2}\textit{Department of Structural Mechanics, University of Pavia, Italy}
E-mail: marco.pingaro@unipv.it, paolo.venini@unipv.it

\textbf{Keywords:} Mixed FEM, Viscoelasticity, Dynamics.

\textbf{SUMMARY.} We present an innovative method for the analysis of viscoelastic plane systems based on a truly-mixed Hellinger-Reissner variational principle, wherein stresses and velocities are the main variables and Lagrange multipliers, respectively. Our discretisation adopts the Arnold–Winther element \cite{1} as to the stress variables along with usual elementwise-linear displacements. An extension to the dynamic case is also introduced and discussed.

\section{INTRODUCTION}

Many engineering materials exhibit creep, i.e. a progressive deformation increment at constant stress, and relaxation that is dually a stress decrement that is experienced at constant strain. Viscoelasticity is the natural framework wherein such behaviors may be soundly modeled and investigated and among the many contributions available for that purpose \cite{2,3} are worth mentioning. From the standpoint of the adopted variational formulation, we shall move from the approach proposed in \cite{4} that introduces a stress-velocity formulations in a truly-mixed setting wherein the symmetry of the stress tensor is imposed weakly. In the present context, the capabilities of the Arnold-Winther stress element are exploited \cite{1} that is the only one in the literature to allow for a continuous stress interpolation with nodal stress values. Furthermore, an extension to the dynamic regime is presented that is capable of including inertial effects in the viscoelastic analysis.

\section{GOVERNING RELATIONS - STRONG FORM}

Reference is made to Figure 1 for the standard viscoelastic solid model that is the basis for the continuum model to be developed hereafter. As usual when adopting Hellinger–Reissner variational principles, compliance tensors relating strains to stresses are introduced that allow one to write

\begin{equation}
\begin{cases}
A_0^{E} \varepsilon_0 + A_0^{V} \sigma_0 = \varepsilon(\dot{u}) \\
A_1^{E} \varepsilon_1 = \varepsilon(\dot{u})
\end{cases}
\end{equation}

where $A_0^{E}$ and $A_0^{V}$ are the elastic and viscous compliance tensors of the viscoelastic component whereas $A_1^{E}$ is the elastic compliance tensor that is in parallel with the viscoelastic one. One should notice that we are using a stress–velocity formulation that presents several advantages over more classical stress–displacement approaches, including the ease with which dynamics effects may be considered in the analysis. Therefore, compatibility relations are written in terms of strain velocities as

\begin{equation}
\begin{cases}
A_0^{E} \varepsilon_0 + A_0^{V} \sigma_0 = \varepsilon(\dot{u}) \\
A_1^{E} \varepsilon_1 = \varepsilon(\dot{u})
\end{cases}
\end{equation}
whereas the dynamic equilibrium reads
\[-\rho \ddot{u} + \text{div } \mathbf{\sigma} = -\rho g. \tag{3}\]

Figure 1: Standard solid phenomenological model

3 The truly–mixed variational formulation

By observing that the total stress \( \mathbf{\sigma} \) may be additively decomposed as \( \mathbf{\sigma} = \mathbf{\sigma}_0 + \mathbf{\sigma}_1 \), the continuous variational formulation of the problem at hand may be obtained by testing the two constitutive relations in (1) by two virtual stress fields \( \tau_0, \tau_1 \), and the equilibrium equation by a virtual velocity field \( \dot{\mathbf{v}} \), so as to write:

Find \( (\mathbf{\sigma}_0, \mathbf{\sigma}_1, \dot{\mathbf{u}}) \in H(\text{div}, \Omega) \times H(\text{div}, \Omega) \times L^2(\Omega) \) such that

\[
\begin{align*}
\langle A^0 \dot{\mathbf{\sigma}}, \tau_0 \rangle &+ \langle A^0 \mathbf{\sigma}_0, \tau_0 \rangle + \langle \dot{\mathbf{u}}, \text{div } \tau_0 \rangle = 0 & \forall \tau_0 &\in H(\text{div}, \Omega) \\
\langle A^1 \dot{\mathbf{\sigma}}, \tau_1 \rangle + \langle \dot{\mathbf{u}}, \text{div } \tau_1 \rangle = 0 & \forall \tau_1 &\in H(\text{div}, \Omega) \\
\langle \rho \ddot{\mathbf{u}}, \dot{\mathbf{v}} \rangle + \langle \text{div } \mathbf{\sigma}_0, \dot{\mathbf{v}} \rangle + \langle \text{div } \mathbf{\sigma}_1, \dot{\mathbf{v}} \rangle &= \langle \rho g, \dot{\mathbf{v}} \rangle & \forall \dot{\mathbf{v}} &\in L^2(\Omega)
\end{align*}
\] \tag{4}

In more compact form, one may rewrite the governing system in Equation (4) in matrix-vector notation as usual within the framework of mixed methods, i.e.

\[
\begin{pmatrix}
A^0_E & 0 & 0 \\
0 & A^0_E & 0 \\
0 & 0 & -M
\end{pmatrix}
\begin{pmatrix}
\dot{\mathbf{\sigma}}_0 \\
\dot{\mathbf{\sigma}}_1 \\
\dot{\mathbf{u}}
\end{pmatrix} +
\begin{pmatrix}
A^0_V & 0 & B^T \\
0 & 0 & B^T \\
B & B & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{\sigma}_0 \\
\mathbf{\sigma}_1 \\
\dot{\mathbf{u}}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\rho g
\end{pmatrix}
\] \tag{5}

It should be emphasized that statics and dynamics model differ only in that the mass matrix \( M \) may be neglected or considered in statics or dynamics, respectively. Equation (5) amounts to a system of ordinary algebraic-differential equations (ADE) in the former case, of differential equations in the latter.
3.1 Imposing Dirichlet and Neumann boundary conditions

Within the adopted truly mixed formulations, Dirichlet boundary conditions on the displacement field

\[ \bar{u} = \bar{u}_0 \] on \( \Gamma_u \)

are imposed weakly thanks to the line integral

\[ \int_{\Gamma_u} (\bar{u} \cdot n) \cdot \bar{u} \, ds \] (6)

that appears on the right hand side of Equations (4)\(_1\) and (4)\(_2\); and in fact taking no action on a boundary amounts to imposing a homogeneous condition \( \bar{u} = 0 \) on \( \Gamma_u \). Neumann traction boundary conditions

\[ \sigma \cdot n = \bar{t} \]

are to be imposed strongly. However, when an additive stress decomposition is adopted as is the case herein, i.e. \( \sigma = \sigma_0 + \sigma_1 \), neither \( \sigma_0 \) nor \( \sigma_1 \) are known but their sum. Therefore a Lagrange multiplier approach is adopted to enforce weakly a condition of type \( \sigma_0 + \sigma_1 = \bar{t} \). Therefore, the resulting linear system takes the form

\[
\begin{pmatrix}
A_E^0 & 0 & 0 & 0 \\
0 & A_E^1 & 0 & 0 \\
0 & 0 & -M & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\sigma_0 \\
\sigma_1 \\
\bar{u} \\
\lambda
\end{pmatrix} +
\begin{pmatrix}
A_V^0 & 0 & B^T & B_2^T \\
0 & 0 & B^T & B_2^T \\
B & B & 0 & 0 \\
B_2 & B_2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\sigma_0 \\
\sigma_1 \\
\bar{u} \\
\lambda
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\rho g \\
t
\end{pmatrix}
\] (7)

4 The Arnold-Winther element

4.1 Basics of the finite element

The triangular Arnold-Winther finite element used in this paper is the lowest–order of the family of finite elements introduced in the pioneering paper [1]. Figure 2 shows the relevant degrees of freedom that may be listed as follows. As to the stresses, one should notice that the symmetry of the stress tensor is imposed strongly so that the components to be approximated are \( \sigma_{11}, \sigma_{22}, \sigma_{12} \) and one ends up with 24 degrees of freedom:

- the 3 components of the stress tensor \( \sigma_{11}, \sigma_{22}, \sigma_{12} \) at each vertex of the triangle (9 dofs);
- the moments of order zero and one of the traction vector \( \sigma \cdot n \) along each edge of the triangle (12 dofs);
- the average of the the components of the stress tensor over the triangle, i.e. \( \int_T \sigma_{11}, \int_T \sigma_{22}, \int_T \sigma_{12} \), (3 dofs).

4.2 Implementation details

No doubt that implementing the Arnold-Winther finite element represents a severe challenge, especially if the goal of minimizing the memory storage is pursued as it should if one recalls that this element is far more expensive than more conventional \((u, p)\)-like elements and other \( H(\text{div}) \) elements such as the one of Johnson and Mercier [5]. A possible implementation of the Arnold-Winther finite element is proposed in [6] that exploits indirect evaluations of the relevant stiffness matrices. Within the present paper, a different semi-analytical approach has been followed [7] inspired by classical
isoparametric elements according to which stress shape functions are first computed analytically on a parent triangular domain. Though not being isoparametric, the Arnold–Winther element enjoys the well-known Piola transformation property:

\[ \sigma(\hat{x}) = B \hat{\sigma}(\hat{x}) B^T \]

that allows to compute the stiffness matrix in any actual configuration.

5 Numerical Studies

The framework presented above is validated hereafter by means of numerical investigations that are concerned with creep and relaxation in both the static and dynamic regimes.

5.1 Relaxation Test

A relaxation test is first performed on a rectangular $8 \times 4$ specimen that is subjected to an imposed displacement field in the direction of its longer side. Since the kinematic variables of our model are the velocities, a constant (in time) displacement field is applied by means of a velocity field that is modulated in time by the function (see Figure 3)

\[
f(t) = \frac{1}{t_2 - t_1/2 + \tau} \cdot \begin{cases} 
  t/t_1 & \text{if } t_1 \leq t \\
  1 & \text{if } t_1 < t \leq t_2 \\
  \exp \left( \frac{t_2 - t}{\tau} \right) & \text{if } t_2 < t
\end{cases}
\]

that has unit integral over the time interval $[0, \infty)$. As to the viscoelastic material, an isotropic behavior is considered for both the elastic and viscous phases that are fully described by the moduli reported in Table 1
5.2 Shear tests

An application of practical interest of the proposed approach is the analysis and design of rubber devices or base isolators for aseismic purposes. Such an investigation under the hypothesis of a purely elastic incompressible behavior has been presented in [8] where the Jonhson-Mercier element was used for analysis and optimal design purposes. As a first extension of such an analysis, the viscoelastic response of a square specimen subjected to pure shear deformation is presented hereafter in both the static and dynamic case. Figures 5 and 7 show the deformed shaped and the evolution of the shear stress value attained at centroidal point of the specimen in the static and dynamic case, respectively. To assess the capability of the proposed method to capture the inertial effects Figures 6 and 8 are eventually presented that show a few snapshots of the shear stress for different time instants. The static maps are self-similar as they should and the only varying quantity is the shear intensity; the dynamic maps conversely show a clear diffusive pattern wherein the perturbation represented by the imposed displacement on top of the specimen is progressively diffused within the entire structure.

6 CONCLUSIONS

A truly-mixed formulation for the analysis of viscoelastic media has been presented whose main variables are the stresses and the velocities. By adopting velocities rather than displacements it is shown that inertial effects may be included rather straightforwardly into the formulation by plugging a consistent mass matrix into one block of the mixed matrices that naturally arise in the procedure. As to the discretization, the Arnold-Wither mixed element is introduced and its unique continuity interpolation properties exploited within a few static and dynamic numerical examples that are presented to validate the framework.

References


Figure 4: Relaxation test: (L) deformed shape - (R) stress evolution


Figure 5: Static shear test: (L) deformed shape - (R) stress evolution

Figure 6: Static shear test: Evolution of the shear-stress wrt time
Figure 7: Dynamic shear test: (L) deformed shape - (R) stress evolution

Figure 8: Dynamic shear test: Evolution of the shear-stress wrt time